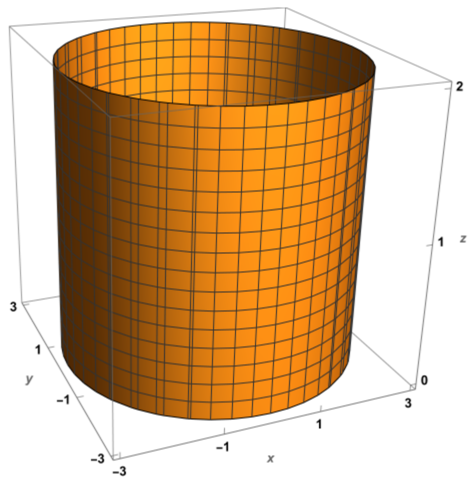
Multivariable Calculus

Quiz 14 SOLUTIONS

1) Compute the scalar surface integral $\iint_{\mathcal{S}} \left(x^2y + z^2\right)\, dS$ where $\mathcal{S}$ is the open cylinder $x^2 + y^2 = 9$ in the range $0 \le z \le 2$.

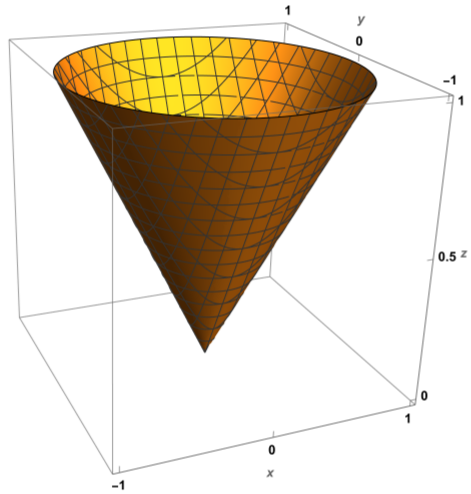


Solution: We can parametrize the cylinder by $\boldsymbol{r}(\theta, z) = \langle 3\cos(\theta),\ 3\sin(\theta), z\rangle$ with $-\pi \le \theta < \pi$ and $0 \le z \le 2$. Since this is a scalar surface integral, the order we cross the partials is irrelevant

$$\boldsymbol{r}_z(\theta, z) \times \boldsymbol{r}_\theta(\theta, z) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 0 & 0 & 1 \\ -3\sin(\theta) & 3\cos(\theta) & 0 \end{vmatrix}$$
$$= \langle -3\cos(\theta),\ -3\sin(\theta),\ 0\rangle$$
$$|\boldsymbol{r}_z(\theta, z) \times \boldsymbol{r}_\theta(\theta, z)| = 3$$
$$\iint_{\mathcal{S}} \left(x^2y + z^2\right)\, dS = \int_{-\pi}^{\pi} \int_0^2 \left(27\cos^2(\theta)\sin(\theta) + z^2\right) dz\, d\theta$$
$$= 54 \int_{-\pi}^{\pi} \cos^2(\theta)\sin(\theta) d\theta + 2\pi \int_0^2 z^2\, dz$$
$$= 0 + \frac{16\pi}{3} = \frac{16\pi}{3}$$

TURN OVER

2) Compute the flux integral $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S}$ for the vector field $\boldsymbol{F}(x, y, z) = \langle x,\ y,\ z^4 \rangle$ where $\mathcal{S}$ is the open cone $z = \sqrt{x^2 + y^2}$ in the range $0 \leq z \leq 1$ with the upward orientation.



Solution: We can parametrize the cone by $\boldsymbol{r}(r, \theta) = \langle r\cos(\theta),\ r\sin(\theta), r\rangle$ with $0 \leq \theta < 2\pi$ and $0 \leq r \leq 1$. To get an upward point normal, we need

$$\begin{aligned}
\boldsymbol{r}_r(r,\theta) \times \boldsymbol{r}_\theta(r,\theta) &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \cos(\theta) & \sin(\theta) & 1 \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} \\
&= \langle -r\cos(\theta),\ -r\sin(\theta),\ r \rangle \\
\boldsymbol{F}(\boldsymbol{r}(r,\theta)) &= \langle r\cos(\theta),\ r\sin(\theta),\ r^4 \rangle \\
\boldsymbol{F}(\boldsymbol{r}(r,\theta)) \cdot (\boldsymbol{r}_r(r,\theta) \times \boldsymbol{r}_\theta(r,\theta)) &= -r^2 + r^5
\end{aligned}$$

$$\begin{aligned}
\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} &= \int_0^{2\pi} \int_0^1 \left(r^5 - r^2\right)\, dr d\theta \\
&= 2\pi \left(\frac{1}{6}r^6 - \frac{1}{3}r^3\right)\Bigg|_0^1 \\
&= -\frac{\pi}{3}
\end{aligned}$$